From the morphism between QFTs to topological Wick rotation III

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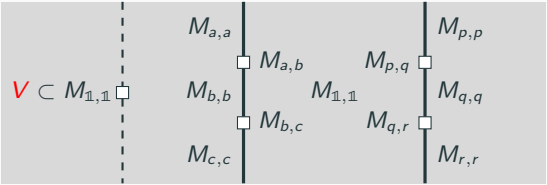
References: K.-Zheng:1705.01087, 1905.04924, 1912.01760, 2011.02859, K.-Wen-Zheng:2108.08835
K.-Lan-Wen-Zhang-Zheng:2003.08898, 2005.14178

Summary of my last talk

The principle of 'bulk=Z(boundary)' applies to an n+1D topological order C^{n+1} with a gapped boundary Kitaev-K.1104.5047,K.:1307.8244 and also to that with a gapless boundary A^n. K.-Wen-Zheng:1502.01690, 1702.00673



- When $n = 2$, an anomaly-free 2+1D topological order C^{2+1} can be characterized by a pair $(\mathcal{C}, c)$, where $\mathcal{C}$ is the modular tensor category (MTC) of anyons and c is the chiral central charge. We should be able to find the categorical characterization of a gapless boundary A^n by solving the mathematical equation $\mathfrak{Z}_1(?) = \mathcal{C}$ or finding the square roots $? = \sqrt{\mathcal{C}}$.



(11) The observables on the 1+1D worldsheet form the following set of data:

$$\Big(\mathcal{S}=\{a,b,c,\cdots\},\ \ \{M_{a,b}\in\mathcal{B}\}_{a,b\in\mathcal{S}},\ \ \{V\hookrightarrow M_{a,a}\}_{a\in\mathcal{S}},\ \ \{M_{b,c}\otimes_V M_{a,b}\xrightarrow{\circ_{a,b,c}} M_{a,c}\}_{a,b,c\in\mathcal{S}}\Big),$$

which, by setting $\mathcal{B}:=\mathrm{Mod}_V$, is precisely a $\mathcal{B}$-enriched category, denoted by ${}^{\mathcal{B}}\mathcal{S}$, such that $\hom_{{}^{\mathcal{B}}\mathcal{S}}(a,b)=M_{a,b}$.

(12) Let $\otimes$ be the horizontal fusion of TDLs. It upgrades ${}^{\mathcal{B}}\mathcal{S}$ to a $\mathcal{B}$-enriched fusion category.

Remark: It turns out that all correlation functions and the OPE among defect fields can be recovered from $(V, {}^{\mathcal{B}}\mathcal{S})$ for $\mathcal{B}=\mathrm{Mod}_V$. Huang:math/0303049,math/0502533, Fuchs-Runkel-Schweigert:2001-2006, Huang-Kirillov-Lepowsky:1406.3420, K.:math/0612255, K-.Runkel:0807.3356, Davydov-K.-Runkel:1307.5956

Theorem (K.-Zheng:1705.01087, 1905.04924, 1912.01760)

The 'rational' gapped/gapless boundaries of a 2+1D topological order $(\mathcal{C}, c)$ can be completely characterized or classified by the triples $(V, \phi, \mathcal{S})$ or $(V, \phi, {}^{\mathcal{B}}\mathcal{S})$, where

1. *for a chiral gapless boundary, V is a rational VOA of central charge c; Huang:math/0502533*
   for a non-chiral gapless boundary, V is a rational full field algebra ($c_L - c_R = c$) K.-Huang:math/0511328;
   for a gapped boundary, $V = \mathbb{C}$. Kitaev-K.:1104.5047
2. *$\mathcal{S}$ is a fusion category equipped with a braided equivalence $\phi : \mathcal{C} \boxtimes \overline{\mathrm{Mod}_V} \xrightarrow{\simeq} \mathfrak{Z}_1(\mathcal{S})$.*

Moreover, the restriction $\phi : \overline{\mathrm{Mod}_V} \xrightarrow{\simeq} \mathcal{B} := \mathcal{C}'_{\mathfrak{Z}_1(\mathcal{S})} = \overline{\mathrm{Mod}_V}$ is a braided equivalence, which determines a $\mathcal{B}$-action on $\mathcal{S}$ and a $\mathcal{B}$-enriched fusion category ${}^{\mathcal{B}}\mathcal{S}$ via the so-called canonical construction. Morrison-Penneys:1701.00567 More precisely, we have $\mathrm{ob}({}^{\mathcal{B}}\mathcal{S}) := \mathrm{ob}(\mathcal{S})$ and $\hom_{{}^{\mathcal{B}}\mathcal{S}}(x, y) = M_{x,y} = [x, y] \in \mathcal{B}$ for $x, y \in \mathcal{S}$, where $[x, y]$ is called internal hom and is defined by the following natural isomorphisms

$$\hom_{\mathcal{S}}(b \triangleright x, y) \simeq \hom_{\mathcal{B}}(b, [x, y]), \qquad \text{for } b \in \mathcal{B}, x, y \in \mathcal{S}.$$

Theorem (K.-Zheng:1704.01447, K.-Yuan-Zhang-Zheng:2104.03121)

The bulk is the center of a boundary, i.e. $\mathcal{C} \simeq \mathfrak{Z}_1({}^{\mathcal{B}}\mathcal{S})$.

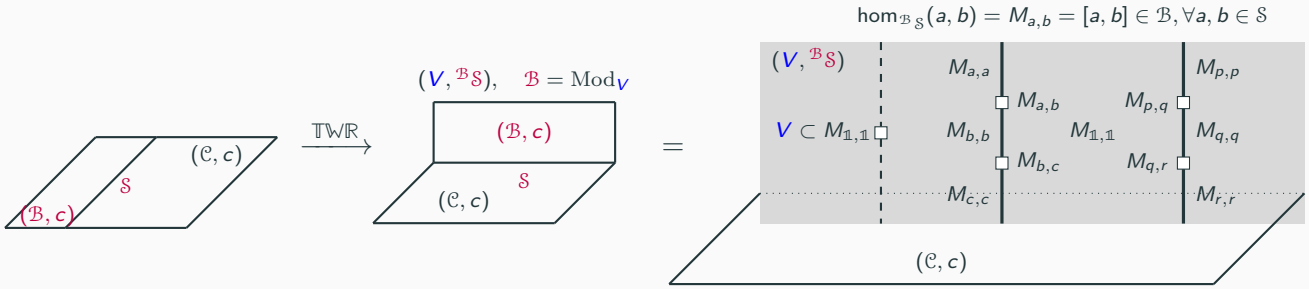
Theorem (K.-Zheng:1705.01087, 1905.04924, 1912.01760)

A gapped/gapless boundary X *of a 2+1D topological order* $(\mathcal{C}, c)$ *can be completely characterized by a pair* $\mathsf{X} = (\mathsf{X}_{\mathrm{lqs}}, \mathsf{X}_{\mathrm{top}})$*, where*

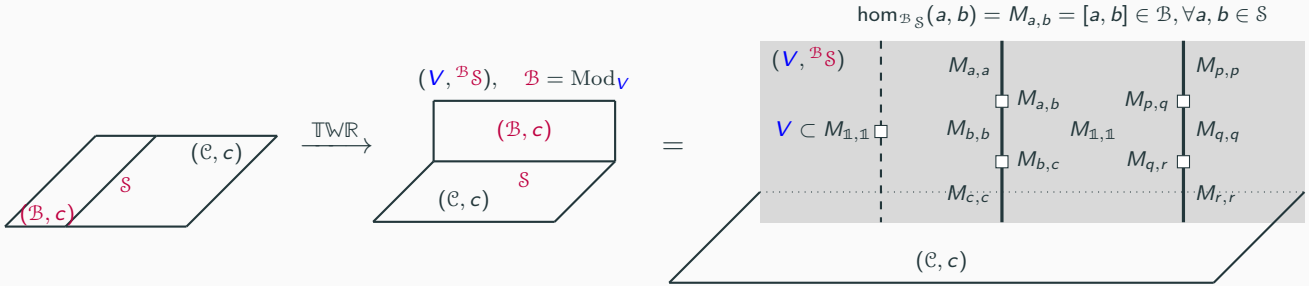
1. $\mathsf{X}_{\mathrm{lqs}} = (V, \phi)$ *is called local quantum symmetry (containing dynamical information);*
2. $\mathsf{X}_{\mathrm{top}} = {}^{\mathcal{B}}\mathcal{S}$ *is called topological skeleton (recall* $\mathcal{B} := \mathcal{C}'_{\mathfrak{Z}_1(\mathcal{S})}$*).*

Moreover, $\mathsf{X}_{\mathrm{top}}$ *can be obtained by topological Wick rotation and* $\mathfrak{Z}_1(\mathsf{X}_{\mathrm{top}}) \simeq \mathcal{C}$*.*





K.-Zheng:1905.04924 "Such a rotation is physically impossible. A picture might be closer to the physical reality is the following one. Imagine that we cut out the topological order $(\mathcal{B}, c)$ along the wall $\mathcal{S}$ by brutal force. It takes the system in the neighborhood of the edge away from a RG fixed point. As time goes by, the edge will undergo a self-healing process by flowing to a new RG fixed point. One of the possible RG fixed point is the gapless edge $(V, {}^{\mathcal{B}}\mathcal{S})$. What makes this picture so amazing is its holographical nature. That is, the information lost through the brutal force cutting in the spatial dimensions can be completely restored in the temporal dimension (as an internal symmetry) by the RG flow!"



Although we recognize its holographical nature from very beginning, we did not name it "topological holography". Note that holography emphasizes the duality between theories in different dimensions. However, we want to emphasize a 'duality' or 'rotation' between "theories" in the same dimension because an $2+1$D topological order is secretly an "$2+0$D theory" from the symmetry point of view.

We choose "rotation" to emphasize that the pair $(\mathcal{B}, \mathcal{S})$ should be viewed as a single structure ${}^{\mathcal{B}}\mathcal{S}$, instead of two independent words in the holographic dictionary. In this talk, I will show why topological Wick rotation is a more powerful tool than topological holography.

Examples of gapless boundaries

Examples of gapped and gapless boundaries:

1. For a non-chiral 2+1D topological order $(\mathfrak{Z}(\mathcal{A}), 0)$, where $\mathcal{A}$ is a fusion category, the pair $(\mathbb{C}, {}^{\mathrm{Vec}}\mathcal{A} = \mathcal{A})$, where $\mathrm{Mod}_{\mathbb{C}} = \mathrm{Vec}$, defines a gapped boundary.
2. For the E_8 invertible 2+1D topological order $(\mathcal{C}, c) = (\mathrm{Vec}, 8)$, the pair $(V_{E_8}, {}^{\mathrm{Vec}}\mathrm{Vec})$, where $\mathrm{Mod}_{V_{E_8}} = \mathrm{Vec}$, defines a non-trivial gapless boundary.
3. Let V be a rational VOA and $\mathcal{C} = \mathrm{Mod}_V$ is MTC.
   - The pair $(V, {}^{\mathcal{C}}\mathcal{C})$ defines a gapless boundary of $(\mathcal{C}, c)$ and $\mathfrak{Z}({}^{\mathcal{C}}\mathcal{C}) \simeq \mathcal{C}$.
   - The triple $(V \otimes_{\mathbb{C}} \overline{V}, {}^{\mathcal{C}\boxtimes\overline{\mathcal{C}}}\mathcal{C})$ defines an anomaly-free 1+1D CFT, where $\mathrm{Mod}_{V\otimes_{\mathbb{C}}\overline{V}} = \mathcal{C} \boxtimes \overline{\mathcal{C}}$, and

$$M_{\mathbb{1}_{\mathcal{C}},\mathbb{1}_{\mathcal{C}}} = [\mathbb{1}_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}] = \oplus_{i\in \mathrm{Irr}(\mathcal{C})}\ i \boxtimes i^*;$$
$$M_{a,b} = [a, b] = \oplus_{i\in \mathrm{Irr}(\mathcal{C})}\ (i \otimes b \otimes a^*) \boxtimes i^*$$

   Moreover, we have $\mathfrak{Z}_1({}^{\mathcal{C}\boxtimes\overline{\mathcal{C}}}\mathcal{C}) = \mathrm{Vec}$.

Examples of chiral gapless edge:

Consider a conformal embedding $V \subsetneq A$ of rational VOA's with the same central charge c. For example:

$$su(m)_n \times su(n)_m \subset su(mn)_1, \quad sp(2m)_n \times sp(2n)_m \subset so(4mn)_1,$$
$$so(m)_n \times so(n)_m \subset so(mn)_1, \quad so(m)_4 \times su(2)_m \subset sp(2m)_1, \cdots .$$



In this case, we have $M_{\mathbb{1},\mathbb{1}} = A \in \mathrm{Mod}_V$, i.e. $V \subsetneq M_{\mathbb{1},\mathbb{1}} = A$.

Examples of non-chiral gapless edges:

- Three modular tensor categories (MTC):
  1. $\mathsf{Is} := \mathrm{Mod}_{V_{\mathsf{Is}}}$ where V_{Is} is the Ising VOA with the central charge $c = \frac{1}{2}$. It has three simple objects $\mathbb{1}, \psi, \sigma$ (i.e. $\mathbb{1} = V_{\mathsf{Is}}$) and the following fusion rules:

$$\psi \otimes \psi = \mathbb{1}, \quad \psi \otimes \sigma = \sigma, \quad \sigma \otimes \sigma = \mathbb{1} \oplus \psi.$$

  2. $\mathfrak{Z}_1(\mathsf{Is}) \simeq \mathsf{Is} \boxtimes \overline{\mathsf{Is}}$. It has 9 simple objects: $\mathbb{1} \boxtimes \mathbb{1}, \mathbb{1} \boxtimes \psi, \mathbb{1} \boxtimes \sigma, \psi \boxtimes \mathbb{1}, \cdots$.
  3. TC = the MTC of toric code. It has four simple objects $1, e, m, f$ and the following fusion rules:

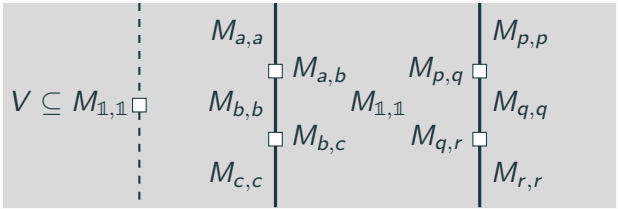
$$e \otimes e = m \otimes m = f \otimes f = 1, \quad m \otimes e = f.$$

- Three non-chiral symmetries (i.e. full field algebras Huang-K.:math/0511328):
  1. $P = V_{\mathsf{Is}} \otimes_{\mathbb{C}} \overline{V_{\mathsf{Is}}} = \mathbb{1} \boxtimes \mathbb{1} \in \mathsf{Is} \boxtimes \overline{\mathsf{Is}} = \mathfrak{Z}_1(\mathsf{Is})$ $\Rightarrow \chi_P = |\chi_0|^2$;
  2. $Q = \mathbb{1} \boxtimes \mathbb{1} \oplus \psi \boxtimes \psi \in \mathfrak{Z}_1(\mathsf{Is})$ $\Rightarrow \chi_Q = |\chi_0|^2 + |\chi_{\frac{1}{2}}|^2$
  3. $R = \mathbb{1} \boxtimes \mathbb{1} \oplus \psi \boxtimes \psi \oplus \sigma \boxtimes \sigma \in \mathfrak{Z}_1(\mathsf{Is})$ $\Rightarrow \chi_R = |\chi_0|^2 + |\chi_{\frac{1}{2}}|^2 + |\chi_{\frac{1}{16}}|^2$

  We have $P \lneq Q \lneq R$.

- P, Q, R are condensable algebras in $\mathfrak{Z}_1(\mathsf{Is})$ and R is a Lagrangian algebra.
  1. $\phi_P : \mathrm{Mod}_P = (\mathfrak{Z}_1(\mathsf{Is}))^0_P \xrightarrow{\simeq} \mathfrak{Z}_1(\mathsf{Is})$: condensing P gives $\mathfrak{Z}_1(\mathsf{Is})$;
  2. $\phi_Q : \mathrm{Mod}_Q = (\mathfrak{Z}_1(\mathsf{Is}))^0_Q \xrightarrow{\simeq} \mathsf{TC}$: condensing Q in $\mathfrak{Z}_1(\mathsf{Is})$ gives toric code TC Bais-Slingerland:0808.0627, Chen-Jian-K.-You-Zheng:1903.12334;
  3. $\phi_R : \mathrm{Mod}_R = (\mathfrak{Z}_1(\mathsf{Is}))^0_R \xrightarrow{\simeq} \mathrm{Vec}$: condensing R in $\mathfrak{Z}_1(\mathsf{Is})$ gives the trivial phase.

$$P = V_{\mathsf{Is}} \otimes_{\mathbb{C}} \overline{V_{\mathsf{Is}}} = \mathbb{1} \boxtimes \mathbb{1}$$
$$Q = \mathbb{1} \boxtimes \mathbb{1} \oplus \psi \boxtimes \psi$$
$$R = \mathbb{1} \boxtimes \mathbb{1} \oplus \psi \boxtimes \psi \oplus \sigma \boxtimes \sigma$$
$$\phi_P : \mathrm{Mod}_P \xrightarrow{\simeq} \mathfrak{Z}_1(\mathsf{Is})$$
$$\phi_Q : \mathrm{Mod}_Q \xrightarrow{\simeq} \mathsf{TC}$$
$$\phi_R : \mathrm{Mod}_R \xrightarrow{\simeq} \mathrm{Vec}$$



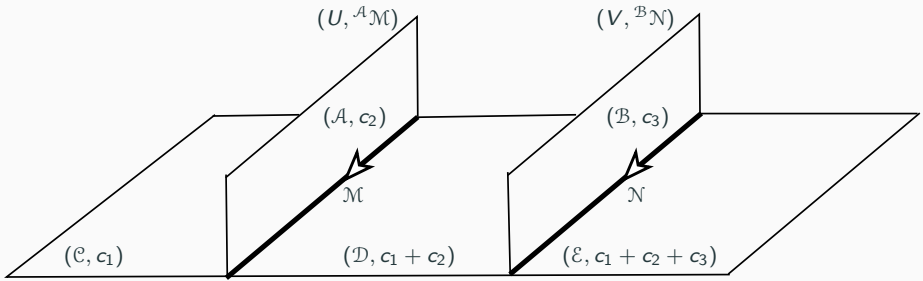
- Four anomaly-free 1+1D gapless quantum liquids defined by triples: i.e. its 2+1D bulk topological order is trivial: $(\mathcal{C}, c) = (\mathrm{Vec}, 0)$ and $\mathfrak{Z}_1({}^{\mathcal{B}}\mathcal{S}) \simeq \mathrm{Vec}$.
  1. $(P, \phi_P, {}^{\mathfrak{Z}_1(\mathsf{Is})}\mathsf{Is})$: in this case $V = P \subsetneq R = M_{\mathbb{1},\mathbb{1}}$;
  2. $(Q, \phi_Q, {}^{\mathsf{TC}}\mathrm{Rep}(\mathbb{Z}_2))$: in this case $V = Q \subsetneq R = M_{\mathbb{1},\mathbb{1}}$;
  3. $(Q, \phi_Q, {}^{\mathsf{TC}}\mathrm{Vec}_{\mathbb{Z}_2})$: in this case $V = Q \subsetneq R = M_{\mathbb{1},\mathbb{1}}$;
  4. $(R, \phi_R, {}^{\mathrm{Vec}}\mathrm{Vec})$: in this case $V = R = M_{\mathbb{1},\mathbb{1}}$.

  In all 4 cases, the space of non-chiral fields living on each 2-cells (i.e. $M_{\mathbb{1},\mathbb{1}}$) is given by the same modular-invariant closed CFT R.

- Gappable gapless edges of 2+1D toric code: $\mathrm{TC} = \mathfrak{Z}_1({}^{\mathfrak{B}}\mathcal{S})$, Chen-Jian-K.-You-Zheng:1903.12334, K.-Zheng:1912.01760
  1. $(P, \phi_P, {}^{\mathfrak{Z}_1(\mathrm{Is})}\mathcal{S})$, where $\mathcal{S} = (\mathfrak{Z}_1(\mathrm{Is}))_Q$ is the fusion category of the right Q-modules in $\mathfrak{Z}_1(\mathrm{Is})$. $\mathcal{S}$ has 6 simple objects $\mathbb{1}, e, m, f, \chi_\pm$, where $\mathbb{1}, e, m, f$ can be identified with 4 anyons in the bulk and $\chi_\pm$ can be identified with two twist defects in the bulk.
  2. $(Q, \phi_Q, {}^{\mathrm{TC}}\mathrm{TC}) =$ the canoncial gapless edge;
  3. $(R, \phi_R, {}^{\mathrm{Vec}}\mathrm{Rep}(\mathbb{Z}_2) = \mathrm{Rep}(\mathbb{Z}_2))$, where $\phi : \mathrm{Mod}_R \xrightarrow{\simeq} \mathrm{Vec}$. Moreover, we have
  $$(R, \phi_R, \mathrm{Rep}(\mathbb{Z}_2)) = \underbrace{(\mathbb{C}, \mathsf{id}, \mathrm{Rep}(\mathbb{Z}_2))}_{\text{the smooth gapped edge}} \boxtimes \underbrace{(R, \phi_R, {}^{\mathrm{Vec}}\mathrm{Vec}).}_{\text{an anomaly-free 2D gapless liquid}}$$
  4. $(R, \phi_R, {}^{\mathrm{Vec}}\mathrm{Vec}_{\mathbb{Z}_2})$:
  $$(R, \phi_R, \mathrm{Vec}_{\mathbb{Z}_2}) = \underbrace{(\mathbb{C}, \mathsf{id}, \mathrm{Vec}_{\mathbb{Z}_2})}_{\text{the rough gapped edge}} \boxtimes \underbrace{(R, \phi_R, {}^{\mathrm{Vec}}\mathrm{Vec})}_{\text{an anomaly-free 2D gapless liquid}}$$
  5. (smooth/rough gapped edge) $\boxtimes$ (any anomaly-free 2D gapless liquid).

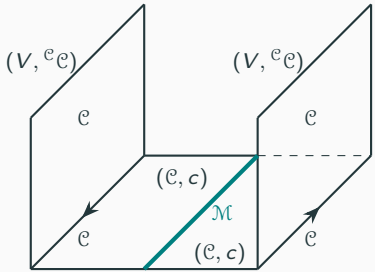
Computing the sandwich

How to compute the fusion of two gapless domain walls? Let U, V be VOAs or full field algebras.

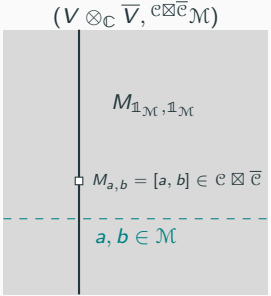


K.-Zheng:1705.01087 $$(U, {}^{\mathcal{A}}\mathcal{M}) \boxtimes_{(\mathcal{D}, c_1+c_2)} (V, {}^{\mathcal{B}}\mathcal{N}) = (U \otimes_{\mathbb{C}} V, {}^{\mathcal{A}\boxtimes\mathcal{B}}\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}),$$

Consider a 2+1D topological order $(\mathcal{C}, c)$ with the canonical gapless boundary $(V, {}^{\mathcal{C}}\mathcal{C})$, where V is a VOA and $\mathcal{C} = \mathrm{Mod}_V$, and a gapped domain wall $\mathcal{M}$: K.-Zheng:1705.01087



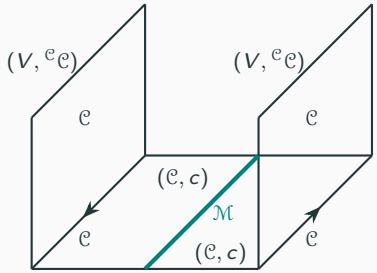
dimensional reduction
to a 1+1D RCFT



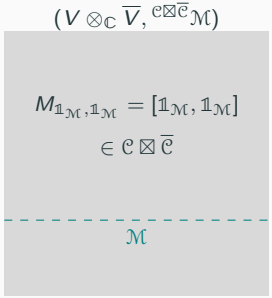
$$(V, {}^{\mathcal{C}}\mathcal{C}) \boxtimes_{(\mathcal{C},c)} (\mathbb{C}, {}^{\mathrm{Vec}}\mathcal{M}) \boxtimes_{(\mathcal{C},c)} (\overline{V}, {}^{\overline{\mathcal{C}}}\mathcal{C}^{\mathrm{rev}}) = (V \otimes_{\mathbb{C}} \overline{V}, {}^{\mathcal{C}\boxtimes\overline{\mathcal{C}}}\mathcal{M}).$$

$M_{\mathbb{1}_{\mathcal{M}},\mathbb{1}_{\mathcal{M}}} = [\mathbb{1}_{\mathcal{M}}, \mathbb{1}_{\mathcal{M}}] \in \mathcal{C} \boxtimes \overline{\mathcal{C}}$ recovers the modular invariant 1+1D rational CFT canonically associated with $\mathcal{M}$. This establishes a one-to-one correspondence:

$$\{\text{gapped domain walls in the 2+1D topological order } (\mathcal{C}, c)\} \overset{1:1}{\longleftrightarrow} \{\text{modular invariant CFT's extending } V \otimes \overline{V}\}$$
$$\mathcal{M} \qquad \mapsto \qquad [\mathbb{1}_{\mathcal{M}}, \mathbb{1}_{\mathcal{M}}] \in \mathcal{C} \boxtimes \overline{\mathcal{C}}$$



dimensional reduction
to a 1+1D RCFT



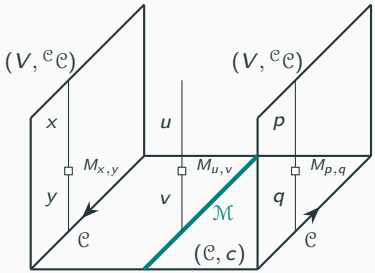
$\{$modular invariant CFT's extending $V \otimes_{\mathbb{C}} \overline{V}\}$

$\xleftrightarrow{1:1}$ $\{$Lagrangian algebras in $\mathcal{C} \boxtimes \overline{\mathcal{C}}\}$ K.-Runkel:0807.3356, Müger:0909.2537

$\xleftrightarrow{1:1}$ $\{$indecomposable $\mathcal{C}$-module categories$\}$ K.-Runkel:0708.1897, Davydov-Müger-Nikshych-Ostrik:1009.2117

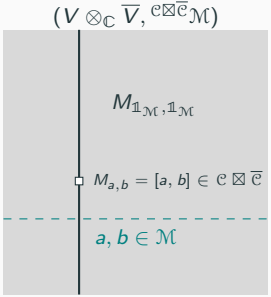
$\xleftrightarrow{1:1}$ $\{$gapped boundaries $\mathcal{M}$ of $(\mathcal{C} \boxtimes \overline{\mathcal{C}}, 0)\}$ Kitaev-K.:1104.5047, K.:1307.8244

$\xleftrightarrow{1:1}$ $\{$gapped domain walls $\mathcal{M}$ in $(\mathcal{C}, c)\}$ folding trick



dimensional reduction

to a 1+1D RCFT

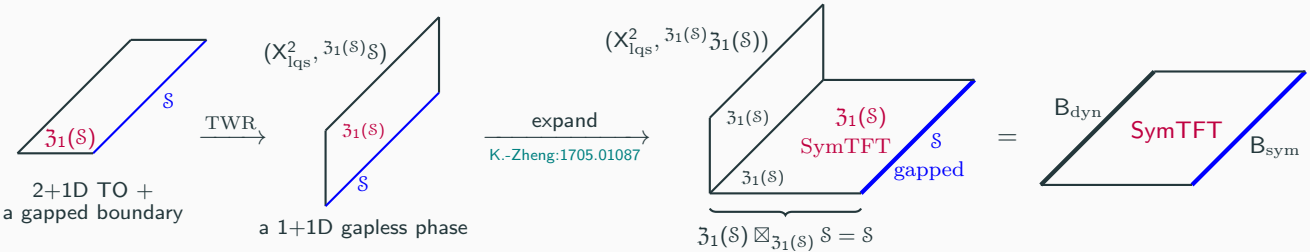


$$(V, {}^{\mathcal{C}}\mathcal{C}) \boxtimes_{(\mathcal{C},c)} (\mathbb{C}, {}^{\mathrm{Vec}}\mathcal{M}) \boxtimes_{(\mathcal{C},c)} (\overline{V}, {}^{\overline{\mathcal{C}}}\mathcal{C}^{\mathrm{rev}}) = (V \otimes_{\mathbb{C}} \overline{V}, {}^{\mathcal{C}\boxtimes\overline{\mathcal{C}}}\mathcal{M})$$

$$M_{x,y} \otimes_{\mathbb{C}} M_{u,v} \otimes_{\mathbb{C}} M_{p,q} \quad \to \quad M_{x\otimes_{\mathcal{C}} u\otimes_{\mathcal{C}}, y\otimes_{\mathcal{C}} v\otimes_{\mathcal{C}} q} \in \mathcal{C} \boxtimes \overline{\mathcal{C}}$$

$$[x, y] \boxtimes [u, v] \boxtimes [p, q] \quad \to \quad [x \otimes_{\mathcal{C}} u\otimes_{\mathcal{C}}, y \otimes_{\mathcal{C}} v \otimes_{\mathcal{C}} q]$$

The idea of sandwich construction or SymTFT (i.e., expanding a 1+1D CFT X^2 to a sandwich such that the fusion category symmetry $\mathcal{S}$ lives on one side of the sandwich and leave the dynamical data on the other side) can now be expressed precisely.



Sandwich expansions of a gapless RCFT $\mathsf{X}^2 = (\mathsf{X}^2_{\text{lqs}}, \mathfrak{Z}_1(\mathcal{S})\mathcal{S})$ are not unique. K.-Wen-Zheng:1502.01690,1507.00503

$$\{\text{sandwich expansions of } \mathsf{X}^2\} \xleftrightarrow{1:1} \{f : \mathcal{T} \to \mathcal{S} \,|\, f \text{ is a monoidal functor; } \mathcal{T} \text{ is a fusion category}\}$$

The sandwich illustrated above is the one associated to the canonical functor $\mathcal{S} \xrightarrow{\text{id}_{\mathcal{S}}} \mathcal{S}$.

Skip unless people ask related questions

Large amount of new questions and open problems arise once we establish the unified mathematical theory of gapped/gapless boundaries of 2+1D topological orders.

1. Reconstructing VOA from a given MTC becomes a very important question.
2. It becomes important to study internal structures lying in each Witt class of modular tensor categories.
3. It becomes very interesting to find irrational solutions to the equation $\mathcal{C} = \mathfrak{Z}_1(?)$. Actually, mathematical solutions of $\mathcal{C} = \mathfrak{Z}_1(?)$ are abundant. However, their physical meanings are unclear.
4. It becomes urgent to develop a mathematical theory of 0+1D domain walls between two gapless boundaries of the same bulk topological order $(\mathcal{C}, c)$. It was partially developed in K.-Zheng:1912.01760, which is far from being complete.

Topological Wick rotation as a new guiding principle for all gapped/gapless quantum liquids

Topological Wick rotation in all dimensions: K.-Zheng:1905.04924,1912.01760,2011.02859

For a (potentially anomalous) quantum liquid $\mathsf{X} = (\mathsf{X}_{\mathrm{lqs}}, \mathsf{X}_{\mathrm{top}})$, its topological skeleton $\mathsf{X}_{\mathrm{top}}$ can be obtained by topological Wick rotation.



We expect "bulk=Z(boundary)" to hold, i.e. $\mathcal{C} \simeq \mathfrak{Z}_1({}^{\mathcal{B}}\mathcal{S})$ K.-Zheng:in preparation.

Remark: In many examples, the same $\mathsf{X}_{\mathrm{top}}$ can be realized by both gapped and gapless phases (depending on $\mathsf{X}_{\mathrm{lqs}}$). A mathematical theory of $\mathsf{X}_{\mathrm{lqs}}$, based on a theory of topological nets in higher dimensions generalizing that of conformal nets in 2D, was proposed and developed in K.-Zheng:2201.05726.

A new type of holographic dualities based on the idea of Topological Wick Rotation K.-Zheng:1705.01087, 1905.04924, 1912.01760, 2011.02859, Ji-Wen:1912.13492, K.-Lan-Wen-Zhang-Zheng:2003.08898, 2005.14178:





an n+1D topological order with a gapped boundary	an nD quantum liquid (SPT/SET/SSB/gapless) with an internal symmetry of finite type
$\mathcal{S}$ is the category of topological defects on the boundary	$\mathcal{S}$ is the category of topological defects { the superselection (charge) sectors of states }
$\mathfrak{Z}_1(\mathcal{S})$ is the category of topological defects in the bulk	$\mathfrak{Z}_1(\mathcal{S})$ is the category of topological sectors of operators { the spaces of non-local operators invariant under LOA}
$\mathfrak{Z}_1(\mathcal{S})$ naturally acts on $\mathcal{S}$	$\mathfrak{Z}_1(\mathcal{S})$ naturally acts on $\mathcal{S}$ leading to an enriched category $\mathfrak{Z}_1(\mathcal{S})_{\mathcal{S}}$

Topological Wick rotation immediately recovers the following well-known classification of 1+1D gapped quantum phases with a finite onsite symmetry G from the classification of the gapped boundaries of 2+1D finite gauge theories.

Theorem (K.-Zheng:2011.02859, K.-Wen-Zheng:2011.02859)

(1) The 1+1D bosonic gapped quantum liquid phases with an onsite symmetry are classified by a triple (G, H, ω), where G is the onsite symmetry defined by a finite group, H is a subgroup of G and $\omega \in H^2(H, U(1))$. *Chen-Gu-Wen:1008.3745, Schuch-Peŕez-García-Cirac:1010.3732*

(2) The topological skeleton of the 1d bosonic gapped phase associated to (G, H, ω) is given by the enriched fusion category ${}^{\mathfrak{Z}_1(\mathrm{Rep}(G))}(\mathfrak{Z}_1(\mathrm{Rep}(G))_{A_{(H,\omega)}})$, where $A_{(H,\omega)}$ denotes the Lagrangian algebra in $\mathfrak{Z}_1(\mathrm{Rep}(G))$ constructed from (H, ω) by Davydov:0908.1250.

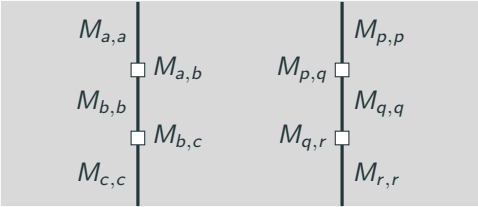
Remark: An independent proposal and proof of above holographic duality in all dimensions based on the idea of categorical symmetry was given in Ji-Wen:1912.13492, K.-Lan-Wen-Zhang-Zheng:2003.08898, 2005.14178. This 3D-2D duality for finite abelian onsite symmetry G was worked out in 1+1D lattice model with onsite G-symmetry in Xu-Zhang:2205.09656, and was rediscovered by high energy theorists in Moradi-Moosavian-Tiwari:2207.10712.

Ising chain:

$$\mathcal{H}_{tot} = \otimes_{i\in\mathbb{Z}} \mathbb{C}_i^2, \qquad H = -\sum_i BX_i - \sum_i JZ_iZ_{i+1},$$

where B and J are the coupling constants, X_i and Z_i are Pauli matrices.

This model has a global onsite $\mathbb{Z}_2$-symmetry defined by the symmetry operator $U = \otimes_i X_i$.



The macroscopic observables on the 1+1D worldsheet form an enriched fusion category ${}^{\mathcal{B}}\mathcal{S}$

Now we work out the SymTO $\mathcal{B}$. In a 1+1D gapped spin chain with an onsite symmetry G,

1. the local quantum symmetry V can be defined by the orbiford VOA $\mathbb{C}^G$;
2. V can be defined by the net of U-symmetric local operators. $\mathrm{Mod}_V^{\mathrm{E}_2}$ is the category of the sectors.

In both cases, $\mathcal{B} = \mathrm{Mod}_V^{\mathrm{E}_2} = \mathfrak{Z}_1(\mathrm{Rep}(\mathbb{Z}_2)) = \mathsf{TC}$. K.-Zheng:2201.05726 Roughly speaking, the objects in $\mathcal{B}$ are the sectors of U-symmetric non-local operators (i.e. the spaces of unconfined non-local operators invariant under the action of symmetric local operators). Ji-Wen:1912.13492, K.-Wen-Zheng:2108.08835

1. $\mathbb{1}$ consists of U-symmetric local operators such as $1, Z_i Z_{i+1}$;
2. m consists of $m_i = \otimes_{k \le i} X_k$ and $U m_i$;
3. e consists of $Z_{i,\infty} = \otimes_{k \ge i} (Z_k Z_{k+1})$;
4. f consists of $m_i Z_{i,\infty}$.

Symmetric phase ($J = 0$): $H = -\sum_i BX_i$

Two simple topological defect lines (TDL) or two charge sectors of states form a fusion category $\mathrm{Rep}(\mathbb{Z}_2)$ (i.e the category of finite dim. $\mathbb{Z}_2$-representations). $X|\pm\rangle = \pm|\pm\rangle$,

$$\{\mathbb{1} \ni |\cdots + + + \cdots\rangle, \quad e \ni |\cdots + + - + + \cdots\rangle\} = \mathrm{Rep}(\mathbb{Z}_2) = \mathcal{S}.$$

Then $M_{a,b}$ is precisely the space of operators that map the sector a of states (or TDL a) to the sector b of states (or TDL b).

$$M_{\mathbb{1},\mathbb{1}} = \mathbb{1} \oplus m \quad \Leftarrow \quad m_j|\cdots + + + \cdots\rangle = |\cdots + + + \cdots\rangle.$$
$$M_{\mathbb{1},e} = M_{e,\mathbb{1}} = e \oplus f, \qquad M_{e,e} = \mathbb{1} \oplus m.$$

$\{M_{a,b}\}_{a,b=\mathbb{1},e}$ form an enriched fusion category ${}^{\mathcal{B}}\mathcal{S} = {}^{\mathrm{TC}}\mathrm{Rep}(\mathbb{Z}_2)$, i.e. $\hom_{{}^{\mathcal{B}}\mathcal{S}}(a, b) = M_{a,b}$.

Theorem (K.-Wen-Zheng:2108.08835)

The Ising chain $H = -\sum_i BX_i$ realizes the trivial 1d $\mathbb{Z}_2$ SPT order, which can be described mathematically by the enriched fusion category ${}^{\mathfrak{Z}_1(\mathrm{Rep}(\mathbb{Z}_2))}\mathrm{Rep}(\mathbb{Z}_2)$.

Symmetry-broken phase ($B = 0$): $H = -\sum_i JZ_iZ_{i+1}$
Two charge sectors of states form a fusion category $\mathrm{Vec}_{\mathbb{Z}_2}$ (i.e the category of $\mathbb{Z}_2$-graded vector spaces).
$Z|\uparrow\rangle = |\uparrow\rangle$, $Z|\downarrow\rangle = -|\downarrow\rangle$

$$\{\mathbb{1} \ni |\cdots\uparrow\uparrow\uparrow\uparrow\cdots\rangle + |\cdots\downarrow\downarrow\downarrow\downarrow\cdots\rangle, \quad m \ni |\cdots\uparrow\uparrow\downarrow\downarrow\cdots\rangle + |\cdots\downarrow\downarrow\uparrow\uparrow\cdots\rangle\} \in \mathrm{Vec}_{\mathbb{Z}_2} = \mathcal{S}.$$

The U-symmetric non-local operator Z_i (or rather $Z_{i,\infty}$) acts on the vacuum $|\cdots\uparrow\uparrow\uparrow\cdots\rangle$ trivially. We immediately obtain:

$$M_{\mathbb{1},\mathbb{1}} = M_{m,m} = \mathbb{1} \oplus e, \qquad M_{\mathbb{1},m} = M_{m,\mathbb{1}} = m \oplus f,$$

Theorem (K.-Wen-Zheng:2108.08835)

The Ising chain $H = -\sum_i JZ_iZ_{i+1}$ realizes a symmetry-breaking phase, which can be described mathematically by the enriched fusion category ${}^{\mathfrak{Z}_1(\mathrm{Rep}(\mathbb{Z}_2))}\mathrm{Vec}_{\mathbb{Z}_2}$.

Theorem (K.-Wen-Zheng:2108.08835)

1. *the symmetric phase of 1+1D Ising chain ($J = 0$):* ${}^{\mathcal{B}}\mathcal{S} = {}^{\mathsf{TC}}\mathrm{Rep}(\mathbb{Z}_2)$;
2. *the symmetry-breaking phase of 1+1D Ising chain ($B = 0$):* ${}^{\mathcal{B}}\mathcal{S} = {}^{\mathsf{TC}}\mathrm{Vec}_{\mathbb{Z}_2}$.

1. $\mathsf{TC} \simeq \mathfrak{Z}_1(\mathrm{Rep}(\mathbb{Z}_2)) \simeq \mathfrak{Z}_1(\mathrm{Vec}_{\mathbb{Z}_2})$ is the category of particles in the bulk of the 2+1D toric code model.
2. $\mathrm{Rep}(\mathbb{Z}_2)$ the category of particles on the smooth boundary of the toric code.
3. $\mathrm{Vec}_{\mathbb{Z}_2}$ is the category of particles on the rough boundary of the toric code.

$$\boxed{\text{2+1D toric code model + gapped boundaries}} \xrightarrow{\text{TWR}} \boxed{\text{1+1D Ising model}}.$$

$$(\mathsf{TC}, \mathrm{Rep}(\mathbb{Z}_2)), \quad (\mathsf{TC}, \mathrm{Vec}_{\mathbb{Z}_2}) \quad \mapsto \quad {}^{\mathsf{TC}}\mathrm{Rep}(\mathbb{Z}_2), \quad {}^{\mathsf{TC}}\mathrm{Vec}_{\mathbb{Z}_2}.$$

Kitaev chain: $\mathcal{H} = \bigotimes_j \mathcal{H}_j$,

$$H = -\mu \sum_j c_j^\dagger c_j - t \sum_j (c_{j+1}^\dagger c_j + c_j^\dagger c_{j+1}) + \Delta \sum_j (c_j c_{j+1} + c_{j+1}^\dagger c_j^\dagger),$$

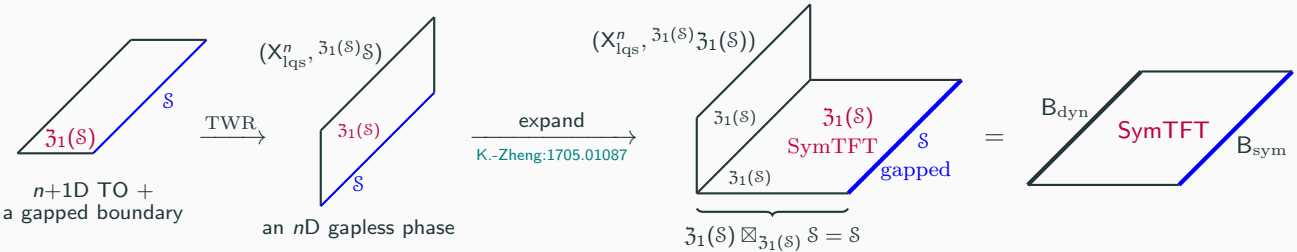
where $\mathcal{H}_j$ is the super vectors space of dimension $1|1$. The fermion parity operator is given by

$$U = \otimes_j \, i\gamma_{j,1}\gamma_{j,2}, \qquad \text{where } \gamma_{j,1} = c_j + c_j^\dagger \text{ and } \gamma_{j,2} = i(c_j - c_j^\dagger).$$

Theorem (K.-Wen-Zheng:2108.08835)

1. $\mu \approx 1$ *and* $t = \Delta = 0$*: the trivial fermionic SPT order given by* $\mathfrak{Z}_1(\mathrm{sVec})_{\mathrm{sVec}}$.
2. $\mu = 0$ *and* $t = -\Delta \approx -1$*: the non-trivial fermionic SPT order given by* $\mathfrak{Z}_1(\mathrm{sVec})^{\mathfrak{Z}_1(\mathrm{sVec})}_{m \leftrightarrow e}\mathrm{sVec}$.

The idea of sandwich construction or SymTFT (i.e., expanding a gapless QFT X^n to a sandwich such that the fusion category symmetry $\mathcal{S}$ lives on one side of the sandwich and leave the dynamical data on the other side) can now be expressed precisely.



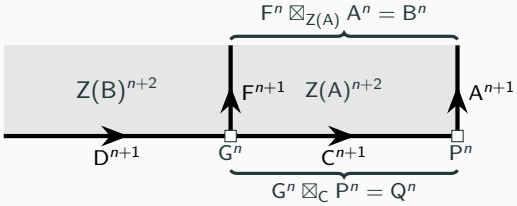
Sandwich expansions of a gapless QFT $\mathsf{X}^n = (\mathsf{X}^n_{\mathrm{lqs}}, \mathfrak{Z}_1(\mathcal{S})\mathcal{S})$ are not unique. K.-Wen-Zheng:1502.01690,1507.00503

$$\{\text{sandwich expansions of } \mathsf{X}^n\} \overset{1:1}{\longleftrightarrow} \{f : \mathcal{T} \to \mathcal{S} \,|\, f \text{ is a monoidal functor; } \mathcal{T} \text{ is a fusion } n\text{-category}\}$$

The sandwich illustrated above is the one associated to the canonical functor $\mathcal{S} \xrightarrow{\mathrm{id}_{\mathcal{S}}} \mathcal{S}$.

Topological Wick rotation further generalizes the notion of a symmetry: K.-Zhang-Zhao-Zheng:2403.07813

The notion of a generalized symmetry can be defined as a morphism between pairs $(\mathcal{A}, \mathcal{P}) \to (\mathcal{B}, \mathcal{Q})$, or equivalently, a morphism between two 'quiches' $(\mathsf{A}^{n+1}, \mathsf{P}^n) \to (\mathsf{B}^{n+1}, \mathsf{Q}^n)$ by a 'generalized sandwich construction' as depicted below:



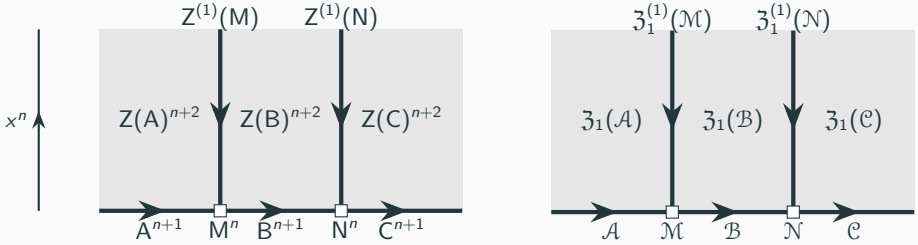
where the generalized 'quiche' $(\mathsf{F}^{n+1}, \mathsf{G}^n)$, which can be viewed as the topological skeleton of a gapped quantum liquid (i.e., SPT/SET/SSB orders), defines the morphism.

The gauging of this further generalized symmetry is automatic and follows from the condensation theory of topological orders.

This holographic dualities were explicitly/implicitly discovered and further studied by different groups of people in different contexts. An incomplete list (no later than 2022):

- Morphisms of QFT & bulk=Z(boundary): Kong-Wen-Zheng:1502.01690, 1702.00673;
- Topological Wick Rotation: K.-Zheng:1705.01087, 1905.04924, 1912.01760, Chen-Jian-K.-You-Zheng:1903.12334, K.-Zheng:2011.02859, K-Wen-Zheng:2108.08835, Xu-Zhang:2205.09656, Lu-Yang:2208.01572
- Categorical Symmetries or SymTO: Ji-Wen:1912.13492, K.-Lan-Wen-Zhang-Zheng:2003.08898,2005.14178, Albert-Aasen-Xu-Ji-Alicea-Preskill:2111.12096, Chatterjee-Wen:2203.03596,2205.06244, Liu-Ji:2208.09101, Chatterjee-Ji-Wen:2212.14432
- 2D Statistical Models: Aasen-Mong-Fendley:2008.08598
- SymTFT and topological holography: Gaiotto-Kulp:2008.05960, Bhardwaj-Lee-Tachikawa:2009.10099, Apruzzi-Bonetti-Etxebarria-Hosseini-Schafer-Nameki:2112.02092, Apruzzi:2203.10063, Moradi-Moosavian-Tiwari:2207.10712, Freed-Moore-Teleman:2209.07471, Kaidi-Ohmori-Zheng:2209.11062, ...
- Other holographic phenomena: Strange correlators: Bal-Williamson-Vanhove-Bultinck-Haegeman-Verstraete:1801.05959; Generalized Kramers-Wannier dualities: Freed-Teleman:1806.00008, Lootens-Delcamp-Ortiz-Verstraete:2112.09091. Modular-covariance of boundary partition functions:Chen-Jian-K.-You-Zheng:1903.12334, Ji-Wen:1905.13279.

Generalizations and Applications



Topological defects in A^{n+1} form a fusion n-category $\mathcal{A}$. The following assignment

$$\mathcal{A} \mapsto \mathfrak{Z}_1(\mathcal{A}) := \mathcal{F}\mathrm{un}_{\mathcal{A}|\mathcal{A}}(\mathcal{A},\mathcal{A}), \qquad {}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}} \mapsto \mathfrak{Z}_1^{(1)}(\mathcal{M}) := \mathcal{F}\mathrm{un}_{\mathcal{A}|\mathcal{B}}(\mathcal{M},\mathcal{M})$$

defines the so-called *center functor* $\mathfrak{Z}_1 : \mathcal{MF}us_n^{\mathrm{ind}} \to {}^{\mathrm{ndg}}\mathcal{BF}us_n^{\mathrm{cl}}$. K.-Wen-Zheng:1502.01690, 1507.00503

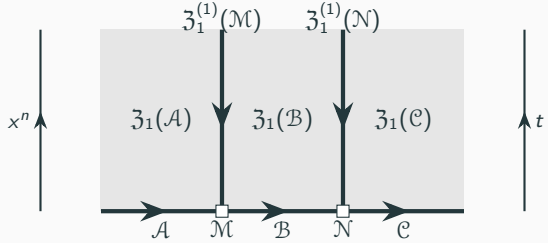
$$\mathfrak{Z}_1^{(1)}(\mathcal{M}) \boxtimes_{\mathfrak{Z}(\mathcal{B})} \mathfrak{Z}_1^{(1)}(\mathcal{N}) \simeq \mathfrak{Z}_1^{(1)}(\mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N}).$$

Topological Wick rotation is functorial!

(1) The holographic dictionary automatically extends to higher codimensional domain walls.

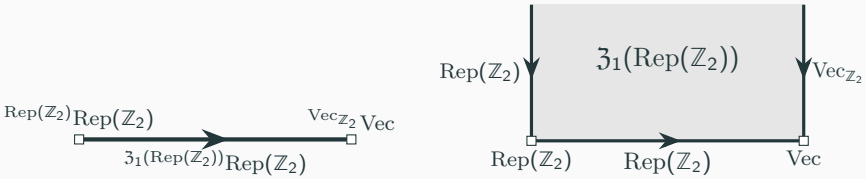
$$(\mathfrak{Z}_1(\mathcal{A}), \mathcal{A}) \mapsto {}^{\mathfrak{Z}_1(\mathcal{A})}\mathcal{A}$$

$$(\mathfrak{Z}_1^{(1)}(\mathcal{M}), \mathcal{M}) \mapsto {}^{\mathfrak{Z}_1^{(1)}(\mathcal{M})}\mathcal{M}$$



This fact has been checked in Gu-Wang-Wen:1004.3835 and in the 1+1D spin chain K.-Wen-Zheng:2108.08835 and in the 1+1D spin chains with a finite abelian symmetry in K.-Zheng:2201.09558

Example: Two boundaries of the symmetric phase in the Ising chain:



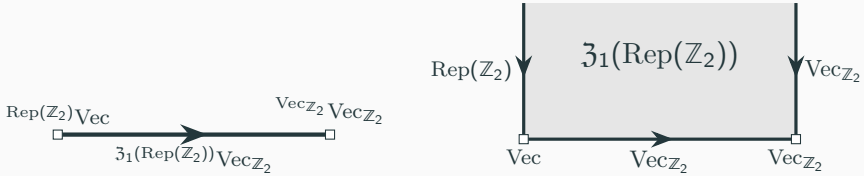
It is clear that the observables in the bulk act on those on the boundary. Therefore, the categorical description of a boundary is necessarily a module over that of the bulk. Indeed, $^{\mathrm{Rep}\mathbb{Z}_2}\mathrm{Rep}\mathbb{Z}_2$ and $^{\mathrm{Vec}_{\mathbb{Z}_2}}\mathrm{Vec}$ are both modules over $^{\mathfrak{Z}_1(\mathrm{Rep}(\mathbb{Z}_2))}\mathrm{Rep}(\mathbb{Z}_2)$, which can be physically realized by two boundaries of Ising chain. K.-Wen-Zheng:2108.08835

Moreover, "bulk=Z(boundary)" holds again K.-Yuan-Zhang-Zheng:2104.03121.

$$\mathfrak{Z}_0(^{\mathrm{Rep}\mathbb{Z}_2}\mathrm{Rep}\mathbb{Z}_2) = {}^{\mathfrak{Z}_1(\mathrm{Rep}(\mathbb{Z}_2))}\mathrm{Rep}(\mathbb{Z}_2) = \mathfrak{Z}_0(^{\mathrm{Vec}_{\mathbb{Z}_2}}\mathrm{Vec}), \quad \mathfrak{Z}_1(^{\mathfrak{Z}_1(\mathrm{Rep}(\mathbb{Z}_2))}\mathrm{Rep}(\mathbb{Z}_2)) = \mathrm{Vec}.$$

We can also see that the center of a center is trivial K.-Yuan-Zhang-Zheng:2104.03121.

Example: Two boundaries of the symmetry-breaking phase in the Ising chain:



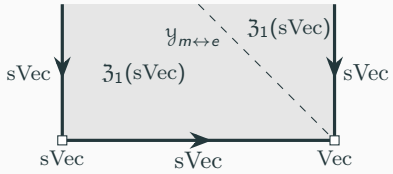
The bulk enriched fusion category $^{\mathfrak{Z}_1(\mathrm{Rep}(\mathbb{Z}_2))}\mathrm{Vec}_{\mathbb{Z}_2}$ has two modules, both of which can be realized physically as the boundaries of the Ising chain.

The principle of bulk=Z(boundary) holds K.-Yuan-Zhang-Zheng:2104.03121, i.e.

$$\mathfrak{Z}_0(^{\mathrm{Rep}(\mathbb{Z}_2)}\mathrm{Vec}) = {}^{\mathfrak{Z}_1(\mathrm{Rep}(\mathbb{Z}_2))}\mathrm{Vec}_{\mathbb{Z}_2} = \mathfrak{Z}_0(^{\mathrm{Vec}_{\mathbb{Z}_2}}\mathrm{Vec}_{\mathbb{Z}_2}), \qquad \mathfrak{Z}_1(^{\mathfrak{Z}_1(\mathrm{Rep}(\mathbb{Z}_2))}\mathrm{Vec}_{\mathbb{Z}_2}) = \mathrm{Vec}$$

Again, the center of a center is trivial.

Example: The boundaries of the two phases in Kitaev chain:



This picture depicts a 2d spatial configuration that can realize two 1+1D gapped phases appeared in 1d Kitaev chain and their boundaries via topological Wick rotation. We use $\mathcal{Y}_{m\leftrightarrow e}$ to denote the invertible domain wall associated to the braided auto-equivalence $\mathfrak{Z}_1(\mathrm{sVec}) \to \mathfrak{Z}_1(\mathrm{sVec})$ defined by $m \leftrightarrow e$. The domain wall $\mathcal{Y}_{m\leftrightarrow e}$ can be mathematically described by the category $\mathcal{F}\mathrm{un}_{\mathrm{sVec}|\mathrm{sVec}}(\mathrm{Vec}, \mathrm{Vec})$ of sVec-sVec-bimodule functors.

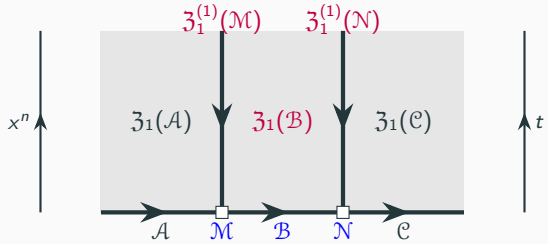
Again the principle of "bulk=Z(boundary)" holds.

$$\mathfrak{Z}_0({}^{\mathrm{sVec}}\mathrm{sVec}) \simeq {}^{\mathfrak{Z}_1(\mathrm{sVec})}\mathrm{sVec}, \qquad \mathfrak{Z}_0({}^{\mathrm{sVec}}\mathrm{Vec}) \simeq {}^{\mathfrak{Z}_1(\mathrm{sVec})}_{m\leftrightarrow e}\mathrm{sVec}.$$

Topological Wick rotation is functorial!

$$(\mathfrak{Z}_1(\mathcal{A}), \mathcal{A}) \mapsto {}^{\mathfrak{Z}_1(\mathcal{A})}\mathcal{A}$$

$$(\mathfrak{Z}_1^{(1)}(\mathcal{M}), \mathcal{M}) \mapsto {}^{\mathfrak{Z}_1^{(1)}(\mathcal{M})}\mathcal{M}$$



(1) The holographic dictionary automatically extends to higher codimensional domain walls.

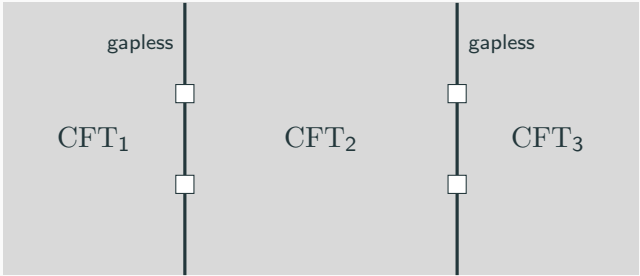
(2) The holographic map is compatible with the fusions of domain walls.

$$({}^{\mathfrak{Z}_1^{(1)}(\mathcal{M})}\mathcal{M}) \boxtimes_{\mathfrak{Z}_1(\mathcal{B})_{\mathcal{B}}} ({}^{\mathfrak{Z}_1^{(1)}(\mathcal{N})}\mathcal{N}) = {}^{\mathfrak{Z}_1^{(1)}(\mathcal{M})\boxtimes_{\mathfrak{Z}_1(\mathcal{B})}\mathfrak{Z}_1^{(1)}(\mathcal{N})}(\mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N}) = {}^{\mathfrak{Z}_1^{(1)}(\mathcal{M}\boxtimes_{\mathcal{B}}\mathcal{N})}(\mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N}).$$

Above formula provides a way to compute the fusion of two gapless domain walls along a gapless liquid.

$$(\mathfrak{Z}_1^{(1)(\mathcal{M})}\mathcal{M}) \boxtimes_{\mathfrak{Z}_1(\mathcal{B})\mathcal{B}} (\mathfrak{Z}_1^{(1)(\mathcal{N})}\mathcal{N}) := \mathfrak{Z}_1^{(1)(\mathcal{M})\boxtimes_{\mathfrak{Z}_1(\mathcal{B})}\mathfrak{Z}_1^{(1)(\mathcal{N})}}(\mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N}) = \mathfrak{Z}_1^{(1)(\mathcal{M}\boxtimes_{\mathcal{B}}\mathcal{N})}(\mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N}).$$

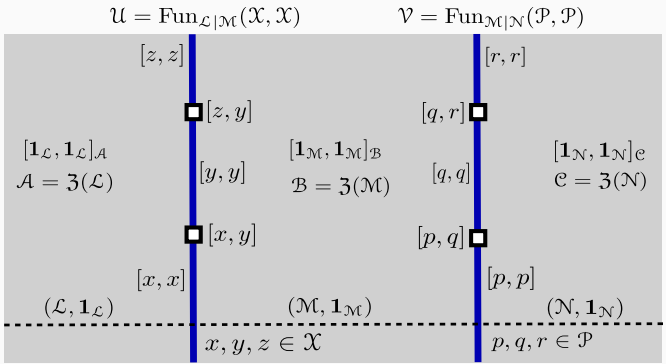
This magical formula makes sense mathematically and can be viewed simply as a mathematical definition of the fusion. However, does it make any sense physically? It is worthwhile to check it in the special case when the bulk QFT is 1+1D rational CFT.



$${}_{\mathcal{L}}\mathcal{X}_{\mathcal{M}}, \quad {}_{\mathcal{M}}\mathcal{P}_{\mathcal{N}}$$

$$\mathcal{U} = \mathfrak{Z}_1^{(1)}(\mathcal{X});$$

$$\mathcal{V} = \mathfrak{Z}_1^{(1)}(\mathcal{P}).$$



The following formula

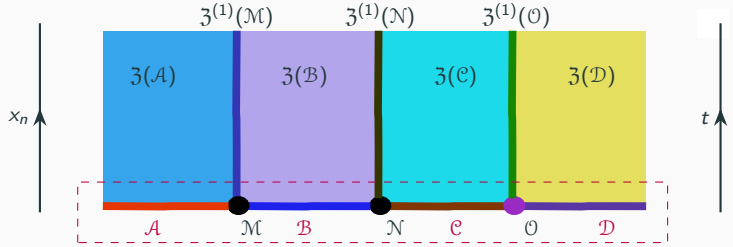
$$\mathfrak{Z}_1^{(1)}(\mathcal{X})\mathcal{X} \boxtimes_{\mathfrak{Z}(\mathcal{M})\mathcal{M}} \mathfrak{Z}_1^{(1)}(\mathcal{P})\mathcal{P} \simeq \mathfrak{Z}_1^{(1)}(\mathcal{X}\boxtimes_{\mathcal{M}}\mathcal{P})(\mathcal{X} \boxtimes_{\mathcal{M}} \mathcal{P}).$$

immediately implies the following fusing formula:

$$[x,y]_{\mathcal{U}} \otimes_{[\mathbf{1}_{\mathcal{M}},\mathbf{1}_{\mathcal{M}}]_{\mathcal{B}}} [p,q]_{\mathcal{V}} \xrightarrow{\simeq} [x \boxtimes_{\mathcal{M}} p, y \boxtimes_{\mathcal{M}} q] \in \mathfrak{Z}_1^{(1)}(\mathcal{X} \boxtimes_{\mathcal{M}} \mathcal{P}), \qquad \forall x,y \in \mathcal{X}, p,q \in \mathcal{P},$$

which was then independently and rigorously proved in K.-Yuan-Zheng:1912.13168.

Topological Wick rotation is functorial.



We denote the category of the topological skeletons of nD anomaly-free quantum liquids by $\mathrm{QL}^n_{\mathrm{top}}$.

Theorem (K.-Zheng:2011.02859)

$$\mathrm{QL}^n \simeq \mathrm{QL}^n_{\mathrm{top}} \simeq \bullet/(n+1)\mathrm{Vec} \simeq \Sigma^{n+1}(\mathbb{C},1),$$

where $(n+1)\mathrm{Vec} = \Sigma^n\mathrm{Vec} = \Sigma^{n+1}\mathbb{C}$ *was defined in Gaiotto-Johnson-Freyd:1905.09566.*

Conclusions and outlooks

In these three talks, I have reviewed four fundamental and general principles in quantum many-body theory or quantum field theory.

1. The principle of "bulk=Z(boundary)" (or boundary-bulk relation).
2. The principle of "bulk=Z(boundary)" is functorial.
3. Topological Wick rotation is a consequence of "bulk=Z(boundary)" and should also be viewed as an independent general principle.
4. Topological Wick rotation is also functorial.

We are witnessing a revolution that provides a unified macroscopic theory of gapped and gapless quantum phases or QFTs, which should automatically include an emerging new theory of phase transitions far beyond Landau's paradigm.